Common-Channel Interference Between Two Frequency-Modulated Signals*

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Summary—*Two frequency-modulated signals received in the same frequency band produce an output from the receiver which is simply a combination of both signals and a beat note whose frequency is modulated in accordance with both signals. The difference of signal strength required for the reduction of crosstalk from the weaker signal is less in frequency modulation than in amplitude modulation, but is the same for different bandwidths of frequency modulation. The required difference can be further reduced by the use of a limiter in the receiver. The beatnote interference remains as background noise fluctuating with the modulation of both signals. This noise is reduced by wide-band frequency modulation. Simple expressions for the detector output in several cases enable the identification of frequency effects which are unavoidably detected as distinguished from amplitude effects which can be removed by a limiter. Common-channel interference is readily tested by oscilloscope patterns. These show the normal operation with or without a limiter, and also the effects of departure from the normal, such as detuning.*

I. Introduction

COMMON-CHANNEL interference is caused by the reception of an undesired signal in the same frequency channel as the desired signal. Such interference is inherently independent of frequency selectivity in the receiver. In amplitude-modulation systems it is independent also of other properties of the receiver, such as the difference between linear and square-law detectors. In frequency-modulation systems, however, such interference is determined by the properties of the receiver and the bandwidth of frequency modulation.

The receiver has a frequency detector which is balanced against amplitude modulation at the unmodulated-carrier frequency of the desired signal. There may or may not be a carrier-amplitude limiter preceding this detector. In response to only one signal, the output of the frequency detector is proportional to the frequency modulation. This is a property of any one of several types of idealized frequency detectors with linear or square-law slope filters and rectifiers.

The amount of common-channel crosstalk interference depends on whether the frequency detector has a linear or a square-law rectifier, unless a perfect limiter is assumed. It is least with a perfect limiter and greatest with square-law rectifiers. It is unaffected by the characteristics of the frequency-modulation system, such as the bandwidth of modulation.

The beatnote interference has the unusual characteristic of simultaneous amplitude and frequency modulation. Its peak amplitude is dependent on the receiver properties but more significantly is affected by some properties of the frequency-modulation system. It is reduced by increasing the bandwidth of frequency modulation. It is further reduced by pre-emphasis and restoration of the higher frequencies of the modulating signal, which incidentally requires a restoring filter after the detector in the receiver.

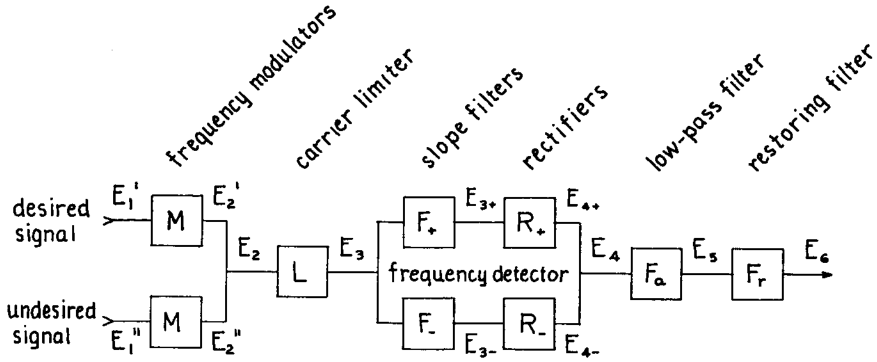

Fig. 1—Frequency-modulation system with interference between two signals in the same frequency band.

The relation between the amplitude and the frequency of the beatnote interference is shown directly by "conical patterns" on the oscilloscope. These are produced by tracing the two-signal output vertically against the difference between the respective modulating signals as the horizontal sweep. This choice of sweep voltage causes the horizontal displacement to be proportional to the beat frequency, so these patterns show directly the effect of any frequency-selective filters following the detector.

All of the relations to be described are based on simple and direct theoretical derivations with the aid of the zero-frequency-carrier concept. There is no assumption as to the waveform of the modulating signals.

The parts of the frequency-modulation system which are essential in the study of common-channel interference are shown in Fig. 1. The audio-frequency modulating voltages are respectively E_1' and E_1'' for the desired and undesired signals. The corresponding

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frequency-modulated-carrier signals are E_2' and E_2''. The composite signal which reaches the receiver is E_2. This may or may not be subjected to the action of a carrier-amplitude limiter before reaching the detector as E_3. The balanced detector comprises a pair of slope filters of opposite slope and a pair of rectifiers. The differential output of the rectifiers is the composite detected signal E_4, including both the desired signal and the interference from the undesired signal.

II. Idealized Balanced Frequency Detectors

An ideal detector for frequency modulation is one which not only reproduces the waveform of frequency modulation but also is unresponsive to amplitude modulation. The latter requirement is partially fulfilled in the balanced frequency detector. This comprises two frequency detectors with a differential output circuit. They convert the frequency modulation into amplitude modulation of opposite polarities. In their differential output appears the signal corresponding to the frequency modulation, but any signal corresponding to original amplitude modulation tends to cancel out. Complete avoidance of response to amplitude modulation would require a perfect limiter preceding the detector.

The elements of the balanced frequency detector are shown in Fig. 1. The slope filters of the two sides have equal response at the center frequency and have opposite slope. The rectifiers are alike but oppositely coupled to the output circuit. These relations assure the nearest approach to cancellation of any output representing incidental amplitude modulation.

The properties of linear slope filters are shown in Fig. 2(a), relative to the unmodulated-carrier frequency in the center. The filter factors are denoted F_+ and F_-. The intercepts at $\pm f_c$ are located arbitrarily for present purposes, but are preferably near the edges of the pass band in practical applications. This location makes the balance least critical and gives sufficient operating range with linear rectifiers.

The use of the linear slope filters with linear rectifiers gives the characteristics of Fig. 2(b).[1] The linear-rectifier properties give rectified voltages equal in magnitude to the envelopes of the voltages from the slope filters. The output of the balanced detector is the difference of the two rectified voltages. In this case, the differential output is proportional to the frequency modulation only between the intercept frequencies $\pm f_c$, as shown in Fig. 2(c).

Square-law rectifiers instead of linear rectifiers modify the frequency characteristics to those of Fig. 2(d). The square-law properties give rectified voltages equal to the square of the magnitudes of the envelopes of the voltages from the slope filters. The differential output, as shown in Fig. 2(e), is proportional to the frequency deviation over an unlimited range. The curvature of the square-law rectifiers cancels out. This type of balanced frequency detector is ideal for theoretical studies, because square-law rectification is most easily formulated in mathematical terms.

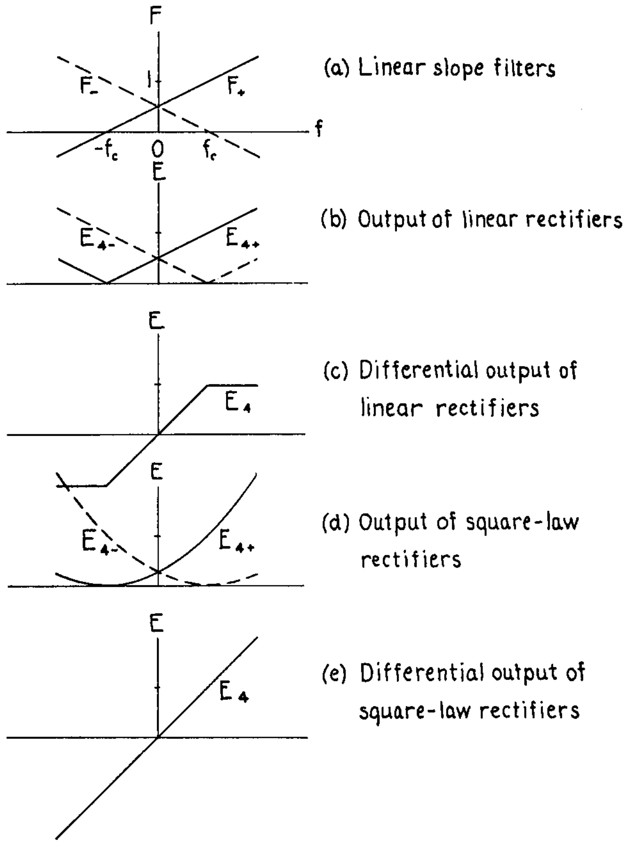

Fig. 2—The essential properties of balanced frequency detectors.

A third type of frequency detector comprises square-law slope filters of the shape of Fig. 2(d) with linear rectifiers. This type also delivers an output like Fig. 2(e) for slow modulation. If the modulation is too rapid, however, this type is found to have a limited range of operation even less in extent than that of linear slope filters with linear rectifiers, shown in Fig. 2(c).

The first type, with linear slope filters and linear rectifiers, is the only one which tolerates a departure from balance without causing distortion of the signal. The other types rely on the balance to cancel the distortion introduced by the square-law characteristics of the individual rectifiers or slope filters.

Linear rectifiers have a practical advantage over square-law rectifiers in that the output signal amplitude varies only half as much with input amplitude. Also the relative response to the amplitude modulation in the composite signal is found to be only half as great with linear rectifiers as with square-law rectifiers.

[1] E. H. Armstrong, "A method of reducing disturbances in radio signaling by a system of frequency modulation," Proc. I.R.E., vol. 24, pp. 689–740; May, 1936. (Balanced frequency detector with linear slope filters and linear rectifiers, his Figs. 5 and 6 compared with Fig. 2 herein.)

From these considerations, it appears that the first type with linear slope filters and linear rectifiers has some advantages over the other types. It is found to operate free of distortion if the frequency modulation is held within the limits of the linear slope in Fig. 2(c), regardless of the waveform of modulation and the marginal sidebands outside of these limits. The most efficient rectifier is the diode peak detector, which best meets the requirement of linearity.

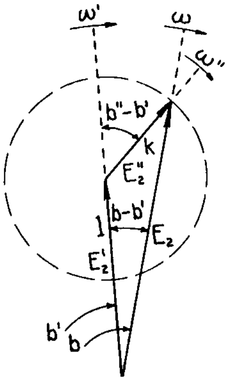

Fig 3—The vector sum of two frequency-modulated signals.

Since the second type with linear slope filters and square-law rectifiers is the simplest for theoretical study, its behavior also is to be described.

If a perfect limiter is assumed preceding the detector, the amplitude effects are removed and it becomes immaterial whether linear or square-law rectifiers are used. Of course, a perfect limiter is not possible in practice, but comparable performance can be obtained with practical limiters of careful design.

III. The Amplitude and Frequency Modulation in the Resultant of Two Signals Superimposed[2]

In response to two signals, there is a difference in operation with and without a limiter. This difference is caused by the amplitude modulation which is present in the composite signal. With a perfect limiter, only the frequency modulation contributes to the detector output.

In the study of the behavior without a limiter, the theoretical derivations do not require the separate expression of the amplitude and frequency modulation. If the limiter is present, however, the frequency modulation does have to be expressed separately.

The superposition of two signals of constant amplitude is shown vectorially in Fig. 3. The unit vector E_2' is the desired signal and the vector E_2'' of length k is the undesired signal. Their frequencies and therefore their phase angles b' and b'' are modulated. The resultant of these two vectors is the composite signal E_2 whose phase angle is b. The average frequency of the composite signal is that of its stronger signal because the relative phase displacement caused by the weaker signal is alternately forward and backward.

Fig. 4 shows the alternating modulations caused by the weaker signal. The fundamental frequency of these modulations is the beat frequency, which is the frequency difference between the two signals. To the extent that the waveforms depart from sine waves, they include also harmonics of the beat frequency.

The relative amount of amplitude modulation is denoted a and its waveform is shown as Fig. 4(a). It is plotted to the scale a/k to show the change of waveform within the same limits and with nearly the same fundamental component. It is noted that this is the shape of the envelope of a carrier accompanied by a single-sideband component.

The alternating phase displacement caused by the weaker signal is shown in Fig. 4(b). It is plotted to the scale b/k, which maintains the same fundamental component. As the amplitude of the weaker signal approaches that of the stronger signal ($k=1$) the phase modulation approaches a saw-tooth waveform. The phase reversal at the instant when the two signals are in opposition appears in the saw-tooth waveform as a phase step of π radians. This case is shown for a value of k approaching unity from the lesser side, so the phase step is negative as required to complete the saw-tooth waveform. This is necessary to assure that the average frequency remains that of the stronger signal.

The corresponding waveforms of frequency modulation are shown in Fig. 4(c), plotted in terms of ω/k

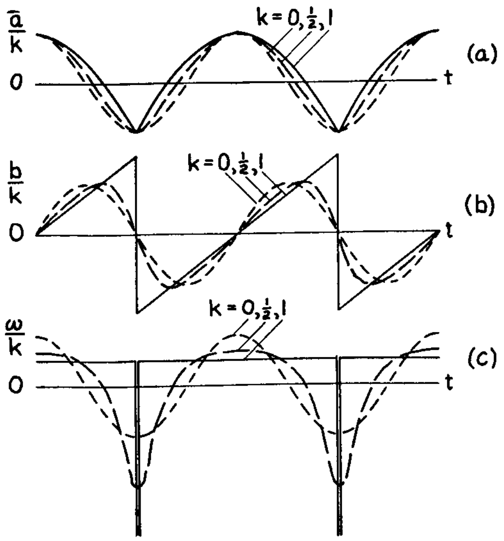

Fig. 4—The beatnote waveforms of
(a) amplitude modulation
(b) phase modulation
(c) frequency modulation.

to maintain the same fundamental component. These are the time derivatives of the phase waveforms. As the signals approach equality, the harmonics approach the fundamental in amplitude and the waveform assumes an impulse shape of very great peak value.

It is these beatnote waveforms of frequency modulation to which the receiver responds if a perfect limiter

[2] Hans Roder, "Noise in frequency modulation," *Electronics*, vol. 10, pp. 22–25, 60, 62, and 64; May, 1937. (The vector resultant of two signals, his Fig. 4 compared with Fig. 3 herein.)

is assumed. Since the average frequency of the composite signal is that of the stronger signal, there is no response proportional to the frequency modulation of the weaker signal. Therefore there is no crosstalk from an undesired signal weaker than the desired signal. Reciprocally, a stronger undesired signal completely masks the desired signal. There is heard only the modulation of the stronger signal, together with the beatnote and its harmonics.

If there is not a perfect limiter assumed, there is some response to the amplitude modulation in the composite signal, Fig. 4(a). The amplitude modulation has not only the beatnote fundamental and harmonic components, but also an increase of its average value caused by the presence of the undesired signal. This is in contrast to the average frequency, which remains unchanged. It is the change of the average amplitude to which is attributed the crosstalk interference from a weaker signal, which is to be described and which occurs only in the absence of a perfect limiter. The beatnote components of the amplitude modulation modify the amplitude and harmonic content of the beatnote output from the frequency detector, in a manner which also remains to be described.

IV. The Response to Two Signals[3,4]

In the derivation of the detector output in response to one or two signals, the conditions in the system are assumed which normally give an output equal to the input modulating signal. Each modulating signal E_1 varies within the limits of ± 1. The resulting frequency modulation of the modulated-carrier signals E_2 is within the limits of $\pm f_c$, the same as the limits on the range of linear operation of the frequency detector with linear slope filters and linear rectifiers, Fig. 2(c). Therefore, the bandwidth of modulation is $2f_c$. The desired signal E_2' has unit carrier amplitude, and the undesired signal has a carrier amplitude k, which is therefore the relative amplitude of the undesired signal.

The simplest expression for the response to two signals is obtained in the case of square-law rectifiers, because it happens that the amplitude and frequency modulation combine to give a beatnote free of harmonics. The output for this case is simply

$$\begin{aligned} E_4 = \; & E_1' && \text{desired signal} \\ & + k^2 E_1'' && \text{crosstalk} \\ & + k(E_1' + E_1'') \cos (b'' - b') && \text{beatnote.} \quad (1) \end{aligned}$$

This output comprises one term which is a replica of the desired-signal modulating voltage E_1' and another which is a replica of the undesired-signal modulating voltage E_1''. The latter is identified as the crosstalk interference, which is recognizable as the undesired signal. The remaining term is the beatnote between the two signals. Its phase is the difference between the progressive phase angles b' of the desired signal and b'' of the undesired signal. Therefore the frequency of the beatnote is modulated in accordance with the difference of the two modulating voltages $(E_1'' - E_1')$. The amplitude of the beatnote is modulated in accordance with the sum of the two modulating voltages. This combination of amplitude and frequency modulation of the beatnote gives it a nondescript character which retains no recognizable qualities of the two signals except their syllabic pulsations.

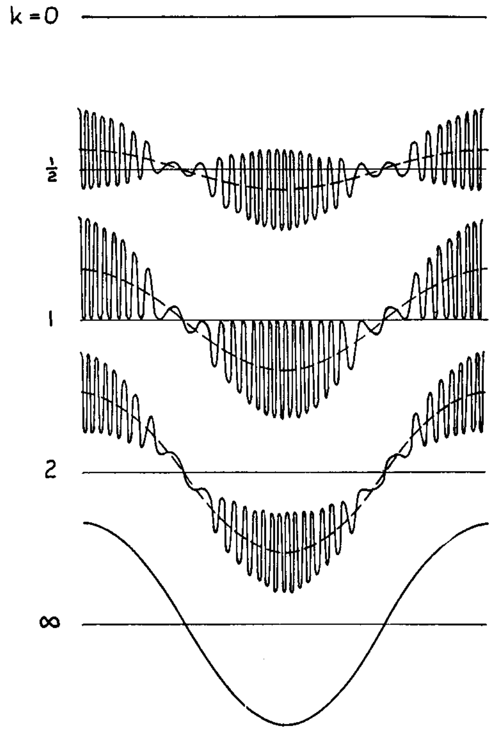

Fig. 5—The output in the case of square-law rectifiers and the desired signal unmodulated showing the crosstalk and beatnote components.

The interference is most noticeable while the desired signal is unmodulated $(E_1' = 0)$, leaving only the crosstalk and beatnote terms. Their amplitude would increase indefinitely with the strength of the undesired signal (k), were it not that an automatic volume control is used. It is assumed that this control holds uniform the mean-square voltage of the composite signal. This has the effect of dividing all terms in (1) by the factor $(1+k^2)$.

On this basis, Fig. 5 shows the interference output in the case of square-law rectifiers, for several different values of the relative strength of the undesired signal. The undesired signal has sinusoidal frequency modulation, the extent of frequency modulation in each direction being $33\pi/2$ times the modulating frequency.

[3] M. G. Crosby, "Frequency modulation propagation characteristics," Proc. I.R.E., vol. 24, pp. 898–913; June, 1936. (Two signals, comprising the same signal received over two different paths.)

[4] M. G. Crosby, "Frequency modulation noise characteristics," Proc. I.R.E., vol. 25, pp. 472–514; April, 1937. (Two signals, comprising a desired signal and noise. The case of a perfect limiter. The separate identification of the desired-signal output and the beatnote interference. The beatnote waveform, his Fig. 4 compared with Fig. 4 herein. The conical pattern, his Fig. 16 compared with Fig. 8 herein.)

Therefore the maximum frequency of the beatnote is $33\pi/2$ times that of the modulation. This large ratio is taken to clarify the difference between the crosstalk of relatively low frequency and the superimposed beatnote of varying frequency. The incommensurate ratio is taken to give an integral number (33) of beatnote cycles during one cycle of modulation. An odd multiple of $\pi/2$ is taken to give a symmetrical waveform of the beatnote during each half cycle of modulation.

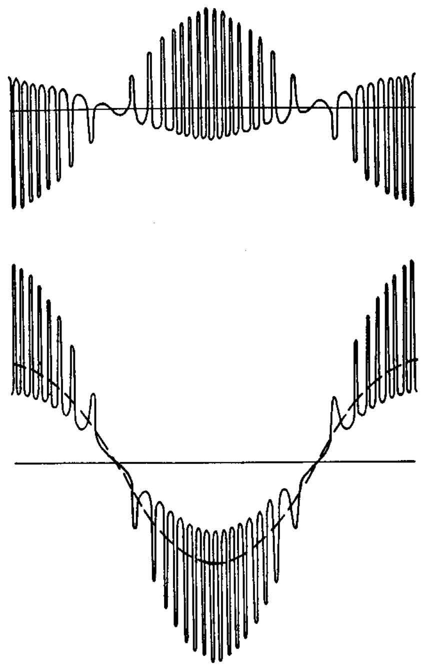

Fig. 6—The output in the case of a perfect limiter showing the unsymmetrical waveform of the beatnote component.

Fig. 5 shows the relative importance of crosstalk and beatnote, depending on the relative signal strength. The beatnote term predominates if the undesired signal is much weaker, and reaches a maximum if the two signals are equal. The crosstalk term increases with the strength of the undesired signal until it exceeds the beatnote term; at the same time, the desired signal is blocked out by the automatic volume control, but this effect is not shown in Fig. 5.

If wide-band frequency modulation is used, the beat frequency during part of the time is so high that it is inaudible and may be filtered out in the audio-frequency amplifier. In this case, the beatnote is diminished during the peaks of modulation, and its wide range of frequency modulation gives it a sizzling or spitting sound.

The corresponding interference with linear rectifiers or a limiter differs in detail but retains the same general characteristics.

In the case of a perfect limiter, it is immaterial which type of rectifier is used in the frequency detector, and also whether the detector is balanced, because the amplitude modulation is removed from the composite signal. If the undesired signal is the weaker ($k<1$), the remaining frequency modulation yields the output

$$\begin{aligned} E_4 = {} & E_1' && \text{desired signal} \\ & + (E_1''-E_1')[k\cos(b''-b') && \text{beatnote} \\ & \quad - k^2\cos 2(b''-b') \\ & \quad + k^3\cos 3(b''-b') - \cdots]. && (2) \end{aligned}$$

There is no crosstalk from the weaker signal, because the average frequency is that of the stronger signal, as noted with reference to Figs. 3 and 4. The beatnote and its harmonics have the waveform of Fig. 4(c). The harmonics are unimportant if there is a substantial difference of signal strength. This case differs from the preceding case in that the beatnote amplitude and frequency are both modulated in step, because both are proportional to the difference of the two modulating voltages $(E''-E')$.

Fig. 6 shows for comparison two of the examples of Fig. 5, but for the case of a perfect limiter. For the undesired signal weaker, only the beatnote appears with its harmonics. The general shape of the waveform is inverted relative to the corresponding example in Fig. 5. With the undesired signal stronger, the crosstalk appears in full strength and the beatnote peaks are inverted. The transition between these two conditions is a critical test for equality of the two signals, if a limiter is used which has nearly ideal properties.

The case of linear rectifiers is the most difficult of analysis because its output terms involve elliptic integrals, as does also its factor of automatic volume control which maintains uniform the average voltage of the composite signal. Also the two signals cause distortion of each other. However, the behavior which is most closely identified with the use of linear rectifiers is easily expressed if one signal is much weaker than the other. It is assumed that the undesired signal is the weaker ($k<1$).

For one approximate expression with linear rectifiers, the only assumption is a difference of signal strength so great that the second and higher powers of k are negligible.

$$\begin{aligned} E_4 = {} & E_1' && \text{desired signal} \\ & + kE_1''\cos(b''-b') && \text{beatnote.} \quad (3) \end{aligned}$$

On these assumptions, the crosstalk from the weaker signal E_1'' is lost, as well as the distortion and masking effects on the desired signal. The harmonics of the beatnote are also lost. There remains only the replica of the desired signal E_1' and the fundamental component of the beatnote interference. It is noted that the amplitude of the beatnote depends on the modulating voltage E_1'' of only the undesired signal, not of both signals as in the preceding cases. The beatnote disappears during interruptions in the modulation of the undesired signal.

For another approximate expression with linear rectifiers, only the third and higher powers of k are neglected, but the desired signal is assumed to be unmodulated ($E_1'=0$) so only the interference remains.

$$E_4 = \frac{k^2}{2} E_1'' \qquad \text{crosstalk}$$
$$+ E_1'' \Big[k \cos (b''-b') \qquad \text{beatnote}$$
$$- \frac{k^2}{2} \cos 2(b''-b') + \cdots \Big]. \qquad (4)$$

Here it appears that all three terms have coefficients midway between the corresponding terms in (1) for square-law rectifiers and those in (2) for a limiter, on the same assumption that the desired signal is unmodulated. The fundamental beatnote term, on these assumptions, is the same in all cases. The crosstalk term is twice as great with square-law rectifiers but absent with a limiter. The second harmonic of the beatnote is twice as great with a limiter but absent with square-law rectifiers.

Since the composite signal comprising both of the frequency-modulated signals has both amplitude and frequency modulation, and since the three cases studied differ only in their response to amplitude modulation, a comparison of these cases enables the effects of amplitude modulation to be identified separately from those of frequency modulation. The desired-signal output is the same in all cases if the undesired signal is much weaker, so the comparisons are based on the interference terms.

The crosstalk comparison for the cases with a limiter or with linear or square-law rectifiers is based on (1), (2), and (4), assuming the desired signal unmodulated and the undesired signal slightly weaker so the third and higher powers of k are negligible. The crosstalk terms in these cases are summarized as follows:

(limiter)	$0\ E_1''$	(2a)
(linear)	$\frac{k^2}{2} E_1''$	(4a)
(square-law)	$k^2\ E_1''$	(1a)
(amplitude effect)	$\frac{k^2}{2} E_1''$.	(5a)

The amplitude effect is the difference between the limiter and linear cases, or one half the difference between the limiter and square-law cases. Since the crosstalk is absent with a perfect limiter, it appears to be caused by the amplitude modulation in the composite signal. It is twice as great with square-law as with linear rectifiers, because the square-law rectifiers are doubly sensitive to amplitude modulation.

The comparison of the beatnote fundamental component for the three cases is based on (1), (2), and (3), assuming only that the undesired signal is sufficiently weak to make negligible the second and higher powers of k.

(limiter)	$(E_1''-E_1')\ k \cos (b''-b')$	(2b)
(linear)	$E_1''\ k \cos (b''-b')$	(3b)
(square-law)	$(E_1''+E_1')\ k \cos (b''-b')$	(1b)
(amplitude effect)	$E_1'\ k \cos (b''-b')$.	(5b)

Again, the effect of amplitude modulation is merely the difference between the limiter and linear cases, or half the difference between the limiter and square-law cases. The significance of this effect remains to be described further on in this discussion. Its cause is not obvious and seems not to be susceptible of simple explanation.

The comparison of the beatnote second-harmonic component is based on (1), (2), and (4), again assuming the desired signal unmodulated and neglecting the third and higher powers of k.

(limiter)	$-\ k^2 E_1'' \cos 2(b''-b')$	(2c)
(linear)	$-\frac{k^2}{2} E_1'' \cos 2(b''-b')$	(4c)
(square-law)	$-\ 0 E_1'' \cos 2(b''-b')$	(1c)
(amplitude effect)	$+\frac{k^2}{2} E_1'' \cos 2(b''-b')$.	(5c)

Here the effect of amplitude modulation is derived in the same manner. It is interesting but this term is of secondary importance in practice.

These three comparisons show that the limiter and square-law cases are the two extremes, while the linear case is intermediate. The limiter case has no crosstalk, while the square-law case has no beatnote harmonics. The linear case is least simple because it has all terms and because the exact expression of the coefficients involves transcendental factors such as elliptic integrals. Each of the three comparisons is based on approximations chosen to show the intermediate value of the coefficient in the linear case. In general, this represents a tendency rather than an exact rule. For comparison with the frequency effects expressed alone in (2), the approximate amplitude effects are summarized as follows:

$$\frac{k^2}{2} E_1'' \qquad \text{crosstalk}$$
$$+ k\ E_1' \cos\ (b''-b') \qquad \text{beatnote}$$
$$+ \frac{k^2}{2} E_1'' \cos 2(b''-b'). \qquad (5)$$

These terms are present, in addition to the frequency terms (2), with linear rectifiers and no limiter. They are doubled by changing to square-law rectifiers and are removed by the insertion of a perfect limiter preceding the detector.

There is a particular significance to the fundamental beatnote term in (5), the only important one of the amplitude terms if the undesired signal is much the weaker. The amplitude of this beatnote term is proportional to the frequency modulation of the desired signal, which is the stronger signal and therefore determines the average frequency of the composite signal. The unbalancing of the frequency detector by the frequency modulation of the stronger signal is a reasonable explanation of the amplitude detection responsible for this beatnote term.

V. The Conical Pattern

While the desired signal and the crosstalk have obvious significance in the detector output, it is difficult to interpret the beatnote as to its interference effect. This is especially true in wide-band frequency modulation, the beatnote being inaudible part of the time. As an aid in testing and interpreting the beatnote interference, the conical pattern is to be described with its various forms and uses.

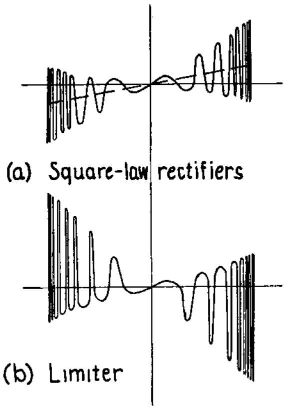

Fig. 7—The output of Figs. 5 and 6 for $k=1/2$ but plotted against the beat frequency to form a conical pattern.

Fig. 7 shows two examples of the conical pattern. They are based on Figs. 5 and 6, but are traced not against time but rather against the modulating voltage of the undesired signal. The desired signal being unmodulated, the beat frequency is proportional to the modulating voltage of the undesired signal. Therefore the output is effectively plotted against the frequency of the beatnote, and this pattern shows the relation between the amplitude and the frequency of the beatnote. Fig. 7 is computed for $k=\frac{1}{2}$, with (a) for square-law rectifiers as in Fig. 5 and (b) for a limiter as in Fig. 6.

Each of the curves in Fig. 7 is called a "conical pattern" because it is the lateral projection of a three-dimensional curve traced on a conical surface by a vector which rotates about the axis of this surface. This vector rotates at the beat frequency, and shifts along the axis at a distance from the center proportional to the beat frequency. In Fig. 7, its amplitude is also proportional to the beat frequency, so its point travels on a conical surface. The conical pattern was originated to show just what appears in Fig. 7, the proportionality between the frequency and amplitude of the beatnote caused by a weak component of noise superimposed on a strong unmodulated carrier of the desired signal.

The conical pattern is most easily shown on an oscilloscope. The output of the receiver and the frequency-modulating voltage of the undesired signal are applied to the respective vertical and horizontal deflecting plates. The modulating frequency and the extent of frequency modulation are usually not critically related, so the pattern appears not as a steady repetitive trace but rather as a trace shifting within the envelope of the pattern. This does not decrease the value of the pattern for showing the relation between frequency and amplitude of the beat note. An oscilloscope having a screen of long persistence is advantageous in observing the envelope rather than the trace itself.

The conical pattern is most distinct if the modulating frequency is much less than the extent of frequency modulation, as in Fig. 7. Otherwise the conical pattern is indistinct, unless there are superimposed a sufficient number of noncoincident traces to determine the envelope. The envelope loses its sharp corners if there is a low-pass filter following the detector, even though this filter has a cutoff frequency slightly higher than the highest frequency of the beatnote. This distortion of the envelope is caused by the failure to retain the sidebands outside the width of frequency modulation, which spread out farther during rapid modulation. If the beatnote has strong harmonics, the envelope is distorted also by the loss of these harmonics.

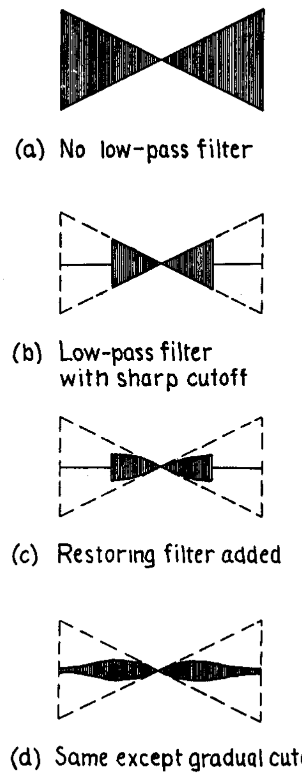

Fig. 8—Conical patterns showing the effect of low-pass filters after the detector.

In Fig. 8 are shown some examples of the conical pattern in its practical applications. In each case, the modulating frequency is sufficiently low to give a sharp outline of the pattern. Fig. 8(a) shows merely the conical envelope without distortion. Fig. 8(b) shows the effect of a sharp-cutoff low-pass filter which passes the beatnote only while the beat frequency is less than the cutoff frequency. This filter is intended to pass as much of the audio range as is needed for adequate reproduction. The conical pattern shows clearly the accompanying reduction of the peak amplitude of the beatnote interference.

If the transmitter employs pre-emphasis of the higher frequencies of modulation, this is compensated by a restoring filter in the receiver.[5] The restoring filter merely attenuates the higher frequencies within the audio range. Its effect on the beatnote output is shown in Fig. 8(c), including a further reduction of the peak amplitude. In practice, the low-pass filter is more likely to have a gradual cutoff which, with the restoring filter, gives a conical pattern of the shape of Fig. 8(d).

The interference effect involves the variation of audibility of the beatnote during the modulation of its frequency. Knowing the characteristics of audition and the frequency scale of the conical pattern enables an estimation of the beatnote interference in terms of its audibility. Such an estimate should take into account also the time during which the beatnote is audible.

The conical patterns of Figs. 7 and 8 are observed with the unmodulated desired signal on the mean frequency of the modulated signal, so the patterns are symmetrical. The center point indicates the occurrence of equality between the frequencies of the two signals and the center frequency of the balanced detector. In the case of a limiter in the receiver, formula (2), it is found that the amplitude of the beatnote is proportional to its frequency, regardless of the tuning relative to the frequency detector, so the crossover point on the envelope always represents equality between the signal frequencies, or zero beat.

Relying on this relationship, Fig. 9 shows the effect of detuning the unmodulated signal. The conical pattern is symmetrical (a) while the unmodulated signal is on the mean frequency of the modulated signal, but departs from symmetry as the unmodulated signal is detuned toward the lower limit of frequency modulation, through (b) to (c). The apex or crossover point of the envelope moves to one edge of the pattern (c) as the unmodulated signal is detuned to the limit of frequency modulation. This gives a critical test for comparing the frequency modulation of one signal against the steady frequency of another. It is most reliable if the modulated signal is weaker than the unmodulated signal, so the conical pattern is nearly horizontal, and the signals differ in strength enough so the requirements on the limiter are not too severe. Such a test is needed for checking the performance of a frequency-modulated signal generator, and would be valuable as a monitor in a transmitter.

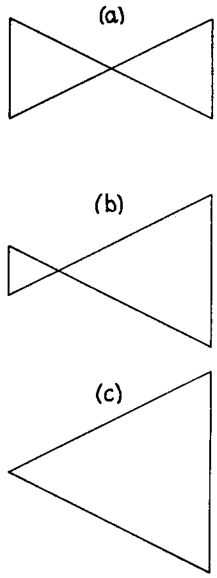

Fig. 9—Conical diagrams showing the effect of detuning an unmodulated signal relative to a frequency-modulated signal.

The detuning of the unmodulated signal during modulation of the other signal gives some indication of what happens to the beatnote amplitude during modulation of both signals. In general, this depends on the limiter and detector properties of the receiver, Fig. 9 being valid only for a receiver with a limiter.

Fig. 10 shows the outer limits of the conical envelope with both signals modulated, in the three cases, of a limiter (a), linear rectifiers (b), or square-law rectifiers (c). These patterns are traced against the difference of the two modulating voltages ($E_1''-E_1'$) so the horizontal displacement is still proportional to the beat frequency. The coefficient of the beatnote term in each of (2), (3), and (1) is used to determine the maximum beatnote amplitude at any beat frequency, both signals being modulated within the same limits (E_1' and E_1'' between -1 and $+1$). The maximum beat frequency occurs with opposite maximum modulation of the signals, so it is double the maximum modulation ($\pm f_c$). It is accompanied by maximum amplitude in the case of a limiter (a) or minimum in the case of square-law rectifiers (c). In the cases without a limiter, it is important that both signals are tuned to the frequency detector. In the case of linear rectifiers (b), the amplitude is modulated only by the undesired signal E_1'', so the peak amplitude of the beat note is the same for all beat frequencies.

The differences among the three cases are present only during simultaneous modulation of both signals,

[5] M. G. Crosby, "The service range of frequency modulation," *RCA Rev.*, vol. 4, pp. 349–371; January, 1940. (The pre-emphasis and restoration of the higher audio frequencies in the modulating signal, his Figs. 4 and 5.)

and are caused by (5b) identified with amplitude detection accompanying the frequency detection. They are associated with the idea that the departure of the stronger signal from the center frequency unbalances the frequency detector, leaving it sensitive to the beatnote amplitude modulation. The greatest amplitude of low-frequency beatnote occurs while both signals

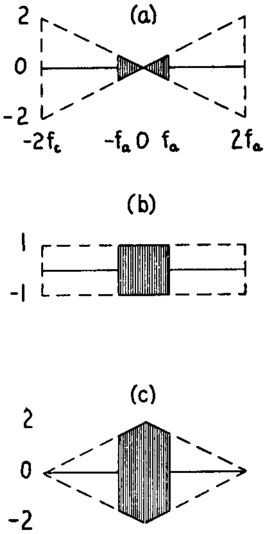

Fig. 10—Conical diagrams showing the maximum amplitude of beatnote during modulation of both signals.
(a) limiter
(b) linear rectifiers
(c) square-law rectifiers.

are modulated in the same sense and to the maximum extent, so the detector is furthest from balance.

The shaded areas in Fig. 10 show the region passed by a low-pass filter similar to that used as a basis for Fig. 8(b). The remaining peak amplitude of the beatnote is least with a limiter (a) and greatest with square-law rectifiers (c). This is in contrast to the beatnote amplitude with one signal unmodulated, which is nearly independent of the limiter and rectifier properties in the receiver. This distinction has little practical significance, because the output of the stronger signal usually obscures such beatnote interference as is caused by its modulation.

VI. The Comparison of Crosstalk and Beatnote Interference Under Various Conditions

Having shown the general characteristics of the output in response to two signals, there remains to present graphically the relative importance of the crosstalk and beatnote interference, depending on the relative strength of the two signals and on the properties of the limiter or of the detector. First, the response of all kinds is to be summarized for the limiter and square-law cases, then the crosstalk output is to be compared for the various cases.

In the case of a perfect limiter, Fig. 11 shows the peak amplitude of all terms relative to that of the desired signal alone. Each term is evaluated during modulation of only one of the two signals. The desired signal (*a*) has its normal value, unless the undesired signal is stronger and therefore completely masks the desired signal. The crosstalk (*b*) is absent if the desired signal is the stronger, but otherwise completely displaces the desired signal. The beatnote fundamental component (*c*) has a maximum value for both signals equal. In general, its relative peak value is equal to the voltage ratio of the weaker signal over the stronger. The beatnote harmonics add to the fundamental, their total peak value becoming indefinitely great for equal signals.

A low-pass filter reduces the peak value of the beatnote fundamental in the ratio of its cutoff frequency over the maximum frequency modulation f_a/f_c, as shown in Fig. 8(b). This is exemplified by Fig. 11(d) for a ratio of 1/5; the low-pass filter may have a sharp cutoff at 15 kilocycles, the upper limit of the audio range, with maximum modulation of 75 kilocycles.

If there is also a restoring filter, of nominal cutoff frequency f_r, and if its cutoff frequency is less than that of the low-pass filter, there is a greater reduction of the beatnote approximately in the ratio f_r/f_c, as shown in Fig. 8(c). This further reduction is indicated in Fig. 11(e) for a ratio of 1/50, the restoring filter having a gradual cutoff at 1.5 kilocycles.

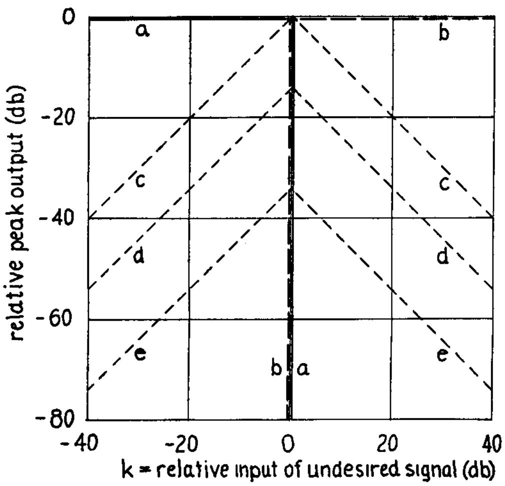

Fig. 11—The relative peak output of the individual components in the case of a perfect limiter.
(a) desired signal
(b) crosstalk
(c) beatnote fundamental component with no low-pass filter
(d) same with added low-pass filter, $f_a = f_c/5$
(e) same with added restoring filter, $f_r = f_c/50$.

The discontinuities in the curves of Fig. 11, at the condition of two equal signals, are caused by the assumption of a perfect limiter. In practice, this assumption fails at this condition, so the curves are rounded off and there is some overlap between the desired signal (a) and the crosstalk (b). These discontinuities are absent in the square-law case and are less severe in the linear case.

In the case of square-law rectifiers, Fig. 12 shows the same relative peak amplitude of the output terms. A perfect automatic amplification control is assumed to hold uniform the mean-square value of the composite signal voltage, to make this case comparable with that of a perfect limiter. The desired signal (*a*) and the crosstalk (*b*) overlap but the stronger signal is favored very much. There is no effect of the stronger signal masking the weaker, as occurs in the limiter, but there is a similar effect of the stronger signal attenuating the weaker through the automatic control. The beatnote curves correspond to those of Fig. 11, (*c*) without any low-pass filter, (*d*) with the low-pass filter, and (*e*) with the restoring filter. In this case, there are no beatnote harmonics.

In each of Figs. 11 and 12, the crosstalk curve (*a*) and the beatnote curve (*b*) add up to unity. This is true as a general rule, on the assumption that there is either a perfect limiter or a perfect automatic volume control using a rectifier of the same type as those in the frequency detector.

The case of linear rectifiers is intermediate in behavior, somewhere between Figs. 11 and 12. All three cases have nearly the same beatnote interference, on the same assumption that only one of the two signals is modulated at a time. The more interesting comparison among the three cases therefore is their crosstalk interference.

Fig. 13 gives a comparison of the crosstalk interference for several cases. With the undesired signal the weaker ($k<1$), the perfect limiter avoids any crosstalk (*a*), and that from linear rectifiers (*b*) is half as great as that from square-law rectifiers (*c*). The curve (*b*) is computed from elliptic integrals and involves the assumption of an automatic control holding uniform the average voltage of the composite signal.

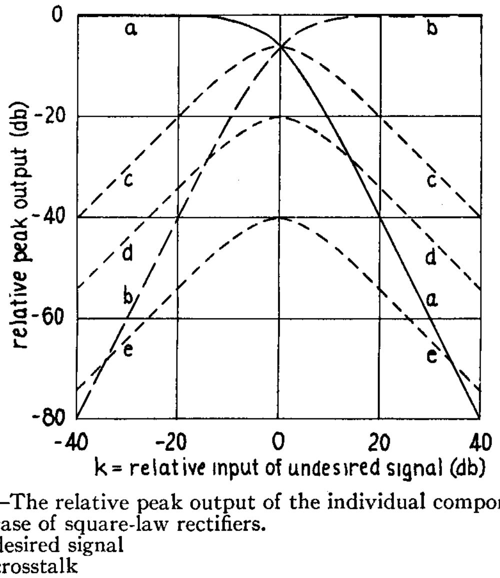

Fig. 12—The relative peak output of the individual components in the case of square-law rectifiers.
(a) desired signal
(b) crosstalk
(c) beatnote with no low-pass filter
(d) same with added low-pass filter, $f_a = f_c/5$
(e) same with added restoring filter, $f_r = f_c/50$.

Practical cases are likely to fall between the case of a perfect limiter (*a*) and that of linear rectifiers (*b*), because nearly linear rectifiers are used with an imperfect limiter, a perfect limiter being impossible of realization. The practical limiter is imperfect in two respects, a failure of limiting action below a certain threshold value, and a departure from uniform output above the threshold value. (Other imperfections such as departure from instantaneous action are here neglected.) If the limiter action is level above the threshold and the desired signal is at least double the threshold voltage, no crosstalk can occur unless the undesired signal exceeds one half the desired-signal voltage. This limit is indicated in Fig. 13(*d*). If instead the limiter has a slope halfway between level and linear, the crosstalk output approaches one half that of the linear case. This is called the "square-root" case, describing its slope of one half, and is indicated in Fig. 13(*e*). Favorable conditions in practice are likely to fall between (*a*) and (*d*).[6]

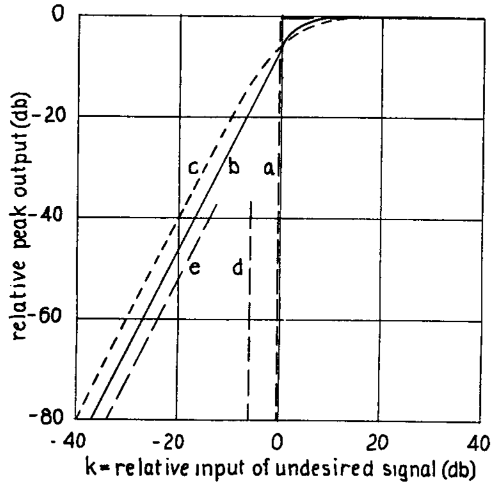

Fig. 13—The relative peak output of the crosstalk component.
(a) perfect limiter
(b) no limiter, linear rectifiers
(c) no limiter, square-law rectifiers
(d) limiter with threshold at 1/2 amplitude of desired signal
(e) square-root limiter and linear rectifiers.

The peak amplitude of the crosstalk is less than that of the beatnote in all cases, if both are less than $(f_r/f_c)^2$ times the peak amplitude of the desired signal. Under the assumptions of maximum frequency modulation, $f_c = 75$ kilocycles, and the restoring filter, $f_r = 1.5$ kilocycles, the beatnote predominates if both interference terms are at least 78 decibels below the desired signal. This relation is based on Fig. 12 for the square-law case, which has the greatest crosstalk. This ratio becomes $2(f_r/f_c)^2$ or -72 decibels in the case of linear rectifiers.

The relative audible interference from the crosstalk and beatnote depends on the kind of program and the requirements of reproduction. For the same peak value, the crosstalk and beatnote are of comparable audibility, but the spitting sound of the beatnote tends to make it the more detrimental. If it is necessary to

[6] I. R. Weir, "Field tests of frequency and amplitude modulation with ultrahigh-frequency waves," *Gen. Elec. Rev.*, vol. 42, pp. 188–191; May, 1939; and pp. 270–273; June, 1939. (The common-channel interference between two signals, his Figs. 9 and 10.)

hold the interference at least 70 or 80 decibels below the desired signal, the beatnote is the determining factor, and the presence or absence of a limiter is unimportant. If more interference is tolerable, the crosstalk is the determining factor and it can be reduced by a limiter.

Increasing the bandwidth of frequency modulation ($2f_c$) for the same bandwidth of the modulating signal (f_a) has no effect on the crosstalk but does enable the reduction of the beatnote output by a low-pass filter after the detector. Such a filter attenuates the beatnote during the time in which the beat frequency is outside the frequency band occupied by the modulating signal.

VII. The Derivation for One Signal in Various Types of Frequency Detectors[7]

The theoretical derivation is simplified by the use of the zero-frequency carrier. This procedure has been shown to yield the practical solution, even in cases of unsymmetrical sidebands as in frequency modulation.[8] The carrier frequency does not appear in the statement of the problem or in its solution.

In this derivation, no specific form of modulating signal is assumed. This avoids the usual limitation to a sinusoidal modulating signal. This generalization actually simplifies the expression of the solution.

The modulating signal, $E_1(t)$ in Fig. 1, has any variation with the time t, subject to some limitation on its bandwidth. (In sound transmission, it is the audio-frequency input signal.) The frequency modulator M is regarded as operating on a carrier of unit amplitude and zero frequency. Therefore the frequency deviation is the same as the frequency f of the modulated signal $E_2(t)$. This frequency deviation is proportional to the modulating signal

$$f(t) = f_c E_1(t) \tag{6}$$

in which f_c is the deviation per unit voltage of E_1.

The three forms of frequency symbols are used interchangeably, whichever fits into the expression at hand:

$$\omega = 2\pi f; \quad p = i\omega = i2\pi f \tag{7}$$

in which ω is the angular frequency and p corresponds to the differential operator D_t with respect to time.

The modulated carrier has a phase angle b related to the frequency modulation as follows:

$$\frac{db}{dt} = \omega = \omega_c E_1 \tag{8}$$

$$b = b_0 + \int_0^t \omega\, dt = b_0 + \omega_c \int_0^t E_1\, dt \tag{9}$$

in which b_0 is the phase angle at $t=0$. The progressive phase angle b henceforth includes both frequency and time variables. The modulated carrier is represented by the vector

$$E_2 = \exp ib = \cos b + i \sin b. \tag{10}$$

There are several types of balanced frequency detectors to be considered, as described with reference to Fig. 2. They differ in the kind of slope filters and the kind of rectifiers. Without loss of generality, the same f_c is taken as the intercept frequency of the slope filters.

The linear slope filter of Fig. 2(a) has the filter factors

$$F_+(f) = \frac{1}{2} + \frac{f}{2f_c} = \frac{f_c + f}{2f_c}; \quad F_-(f) = \frac{1}{2} - \frac{f}{2f_c} = \frac{f_c - f}{2f_c} \tag{11}$$

in which the + and − subscripts denote the two sides of the balanced detector. Each of these factors changes between zero and unity between the intercepts at $\pm f_c$. The fact that the slope continues over negative and positive values of the filter factor denotes the use of a resonant trap in each filter.

The filter factor on each side is expressed in terms of the differential operator and then applied to the modulated voltage E_2 to secure the differentiated voltages E_{3+} and E_{3-} in Fig. 1:

$$F_\pm = \frac{1}{2} \pm \frac{p}{2p_c} = \frac{1}{2} \pm \frac{1}{2p_c} D_t \tag{12}$$

$$E_{3\pm} = \left(\frac{1}{2} \pm \frac{1}{2p_c} D_t\right) E_2 = \frac{1}{2} \exp ib \pm \frac{i}{2p_c} \cdot \frac{db}{dt} \exp ib$$
$$= \left(\frac{1}{2} \pm \frac{1}{2} E_1\right) \exp ib. \tag{13}$$

This shows the amplitude modulation superimposed on the frequency modulation by the slope filters.

This application of the differential operator p is common ground between Heaviside operational methods and Fourier integral methods.[9,10] In this case, it is the same as the transformation

$$E = (R + j\omega L)I = (R + Lp)I = (R + L \cdot D_t)I = RI + L\frac{dI}{dt} \tag{14}$$

in which E is the voltage, R is the resistance, L is the inductance, and I is the current having any variation with the time t, and D_t means d/dt.

The resulting voltages E_{3+} and E_{3-} are delivered to the rectifiers, which are linear rectifiers in this case. Since the unrectified signal is expressed as a modulated zero-frequency carrier, its magnitude is the signal envelope to which the rectifier responds. The magnitude

[7] J. R. Carson and T. C. Fry, "Variable frequency electric circuit theory with application to the theory of frequency modulation," *Bell Sys. Tech. Jour.*, vol. 16, pp. 513–540; October, 1937.

[8] H. A. Wheeler, "The solution of unsymmetrical-sideband problems with the aid of the zero-frequency carrier," Proc. I.R.E., vol. 29, pp. 446–458; August, 1941.

[9] Vannevar Bush, "Operational Circuit Analysis," 1929. See pp. 17–21.

[10] G. A. Campbell and R. M. Foster, "Fourier integrals for practical applications," Bell Telephone System Monograph B-584, September, 1931. Abridgment, *Bell Sys. Tech. Jour.*, vol. 7, pp. 639–707, October 1928. See Table I, No. 208.

of E_{3+} or E_{3-} carries an amplitude modulation which is proportional to the modulating signal E_1.

The linear rectifiers produce output voltages E_{4+} and E_{4-} equal to the envelope amplitudes of the differentiated signals, in this case the magnitude of the voltages E_{3+} and E_{3-}:

$$E_{4\pm} = |E_{3\pm}| = |\tfrac{1}{2} \pm \tfrac{1}{2}E_1| \quad (15)$$

as shown in Fig. 2(b). The output voltage E_4 from the balanced detector is the differential output of the rectifiers:

$$E_4 = E_{4+} - E_{4-} = |\tfrac{1}{2} + \tfrac{1}{2}E_1| - |\tfrac{1}{2} - \tfrac{1}{2}E_1|. \quad (16)$$

This has a range of linear proportionality limited by the intercept frequencies, in which the output is equal to the modulating voltage:

$$-f_c < f < f_c; \quad -1 < E_1 < 1; \quad E_4 = E_1 \quad (17)$$

as shown in Fig. 2(c).

In this derivation, it is notable that the bandwidth between the intercept frequencies $\pm f_c$ need be only sufficient to include the maximum value of the deviation f. The marginal sidebands outside of this width $2f_c$ do not require a further separation of the intercept frequencies but do require a continuation of the linear slope as far out as the sidebands are appreciable.

The next case is that of the same linear slope filters with square-law rectifiers, as shown in Fig. 2(d) and (e). The rectified voltages are taken equal to the square of the signal magnitudes:

$$E_{4\pm} = |E_{3\pm}|^2 = (\tfrac{1}{2} \pm \tfrac{1}{2}E_1)^2 = \tfrac{1}{4} \pm \tfrac{1}{2}E_1 + \tfrac{1}{4}E_1^2. \quad (18)$$

The differential output is simply

$$E_4 = E_{4+} - E_{4-} = E_1. \quad (19)$$

This is valid over an unlimited range, irrespective of the intercept frequencies of the slope filters. Therefore this type of balanced frequency detector is ideal for theoretical purposes.

The remaining case is that of square-law slope filters and linear rectifiers. The slope filters have the form of Fig. 2(d) and the differential output has the form of Fig. 2(e). The filter factors are those of (11) and (12), squared:

$$F_\pm = \left(\frac{1}{2} \pm \frac{f}{2f_c}\right)^2 = \frac{1}{4} \pm \frac{p}{2p_c} + \frac{p^2}{4p_c^2}$$
$$= \frac{1}{4} \pm \frac{1}{2p_c} D_t + \frac{1}{4p_c^2} D_t^2. \quad (20)$$

This factor requires two resonant traps at the intercept frequency in each of the slope filters. Applying this factor to E_2, the differentiated voltages are found to be

$$E_{3\pm} = \left(\frac{1}{4} \pm \frac{1}{2}E_1 + \frac{1}{4}E_1^2 + \frac{1}{4p_c}\frac{dE_1}{dt}\right) \exp ib. \quad (21)$$

The output of each linear rectifier is the magnitude

$$E_{4\pm} = \left|\frac{1}{4} \pm \frac{1}{2}E_1 + \frac{1}{4}E_1^2 + \frac{1}{4p_c}\frac{dE_1}{dt}\right|$$
$$= \left|\left(\frac{1}{2} \pm \frac{1}{2}E_1\right)^2 + \frac{1}{4p_c}\frac{dE_1}{dt}\right|. \quad (22)$$

If the modulating voltage E_1 has only slow variations, the differential output is simply E_1, as in the preceding case. However, such an assumption is not generally justified, in which case the peak value of E_1 must be held less than one, by an amount sufficient to assure the first of the last two terms exceeding the second. It is concluded that this type, because of its double differentiation, gives a linear output which is reliable over a lesser range of modulation than the first type with linear slope filters and linear rectifiers.

The theory of these three types of balanced frequency detectors indicates advantages for the first and second types. The first type, with linear slope filters and linear rectifiers, has the advantage that an undistorted replica of the modulating voltage appears in each of the rectifiers, so a departure from balance leaves no distortion of the output. This type has the disadvantage of operating over only a limited range of frequency modulation. The second type, with linear slope filters and square-law rectifiers, is ideal for theoretical purposes because it has an unlimited range of operation, and because square-law rectifiers are most susceptible of mathematical treatment. It has the practical disadvantages that square-law rectifiers are less efficient. Also they cause distortion in each rectifier so exact balance is required to secure an undistorted output.

These two types of detectors are denoted simply the linear and square-law types, referring to the rectifiers; both types have linear slope filters.

With only one signal and ideal conditions, the presence or absence of a limiter is immaterial, because the signal amplitude is uniform.

VIII. The Derivation For Two Signals in a Square-law Frequency Detector

Each of the two signals has the form of the one signal in the preceding section. The desired and undesired signals are identified by superscripts (′ and ″) as in Fig. 1.

For the study of common-channel interference, these two signals are to be superimposed. The desired signal E_2' has unit amplitude while the undesired signal E_2'' has a relative amplitude k, constant and usually less than 1. The composite signal is

$$E_2 = E_2' + E_2'' = \exp ib' + k \exp ib''. \quad (23)$$

Referring to equations (12) and (13) for the linear filter factors and the form of the differentiated signal, the composite signal becomes

$$E_{3\pm} = (\tfrac{1}{2} \pm \tfrac{1}{2}E_1') \exp ib' + k(\tfrac{1}{2} \pm \tfrac{1}{2}E_1'') \exp ib''$$
$$= [(\tfrac{1}{2} \pm \tfrac{1}{2}E_1') + k(\tfrac{1}{2} \pm \tfrac{1}{2}E_1'') \exp i(b''-b')] \exp ib'. \quad (24)$$

The square-law rectified voltages are the squared magnitudes of the coefficient in brackets [ ], since exp ib' is a unit vector.

$$E_{4\pm} = |E_{3\pm}|^2 = [(\tfrac{1}{2} \pm \tfrac{1}{2}E_1') + k(\tfrac{1}{2} \pm \tfrac{1}{2}E_1'') \cos (b''-b')]^2$$
$$+ [k(\tfrac{1}{2} \pm \tfrac{1}{2}E_1'') \sin (b''-b')]^2$$
$$= (\tfrac{1}{4} \pm \tfrac{1}{2}E_1' + \tfrac{1}{4}E_1'^2) + k^2(\tfrac{1}{4} \pm \tfrac{1}{2}E_1'' + \tfrac{1}{4}E_1''^2)$$
$$+2k(\tfrac{1}{4} \pm \tfrac{1}{4}E_1' \pm \tfrac{1}{4}E'' + \tfrac{1}{4}E_1'E_1'') \cos (b''-b'). \quad (25)$$

In the differential output, all except the ± terms cancel out, leaving simply

$$E_4 = E_{4+} - E_{4-}$$
$$= E_1' \quad \text{desired signal}$$
$$+ k^2E_1'' \quad \text{crosstalk}$$
$$+ k(E_1' + E_1'') \cos (b''-b') \quad \text{beatnote.} \quad (26)$$

As is characteristic of square-law rectifiers, the desired-signal output is unaffected by the presence of the undesired signal. The crosstalk output is free of distortion and proportional to k^2. The relative phase of the two carriers is noncritical.

The beatnote interference is unusual in its properties. It is a sinusoidal wave whose amplitude and frequency are modulated by both signals. Its frequency is

$$f'' - f' = \frac{1}{2\pi} \frac{d}{dt} (b''-b') = f_c(E_1''-E_1'). \quad (27)$$

This is the instantaneous difference of the two frequency deviations, so it is determined by the frequency modulation of both signals. In speech modulation, the beatnote interference is heard as an irregular harsh rasping noise having only syllabic relation to the modulating signals. In wide-band modulation, the beatnote is inaudible some of the time.

Under the influence of automatic volume control (not the same as a limiter), there is manifested a blocking effect of either signal on the other, as shown in Fig. 12. The total power of the two signals is $1+k^2$, and the square-law rectifier response is proportional to the power. Dividing the output voltage (19) by this factor gives the relative amplitudes when subjected to an action which maintains uniform the average power input to the frequency detector.

$$\frac{E_4}{1+k^2} = \frac{1}{1+k^2}E_1' + \frac{k^2}{1+k^2}E_1''$$
$$+ \frac{k}{1+k^2}(E_1'+E_1'') \cos (b'-b''). \quad (28)$$

The curves of Fig. 12 are plotted in terms of peak values of output for E_1' or E_2' having unit peak values. Curve (*a*) is the blocking of the desired modulation by the undesired carrier unmodulated. Curve (*b*) is the reverse effect, the crosstalk from the undesired modulation while the desired carrier is unmodulated.

While the mean or effective value of the beatnote interference is difficult to compare with the desired modulation, its peak value is interesting and can be expressed for some cases. The maximum amplitude of the beatnote occurs when the amplitude coefficient $(E_1'+E_1'')$ in (26) has its maximum value of two, and the frequency coefficient $(E_1''-E_1')$ in (27) is small; this means when both signals have maximum modulation of the same polarity and nearly but not exactly the same amplitude. The relative peak value of beatnote interference, while both signals are modulated, is therefore independent of the bandwidth of modulation, but the relative mean value and average audibility is reduced by wider modulation. This is shown in Fig. 10(c), in which the peak amplitude of the beatnote is plotted against the beat frequency.

If only one signal is modulated, as for the curves of Fig. 12, the peak value of beatnote interference is reduced by wider modulation which increases the beat frequency beyond the range of audibility. The audibility decreases so rapidly above 5 kilocycles, that a low-pass filter may be assumed following the detector with a cutoff frequency f_a of about 5 kilocycles. Such a filter may be present, or may be approximated by a high-audio-frequency attenuator intended to compensate for emphasis in the transmitter. The maximum modulating voltage E_1 on either signal, which produces a beat frequency f_k not exceeding f_a, is found from (27) to be

$$E_{1a} = \pm \frac{f_a}{f_c}. \quad (29)$$

The corresponding peak voltage of the beatnote interference, multiplied by the factor of automatic volume control, is found from (28) to be

$$\frac{k}{1+k^2} \cdot \frac{f_a}{f_c}. \quad (30)$$

This peak value is plotted in Fig. 12 as curves (*c*) and (*d*) for the ratio $f_a/f_c = 1$ and 1/10. This corresponds to a maximum deviation f_c one and ten times the audio frequency f_a.

For laboratory studies, sinusoidal modulation is most convenient. Therefore this case for frequency modulation deserves an explicit solution of the interference waveform. The example chosen is the unmodulated desired signal and sinusoidal modulation of the undesired signal. The modulating wave is

$$E_1'' = m \cos \omega_m t \quad (31)$$

in which m is the modulation factor such that the maximum deviation is mf_c. The resulting frequency deviation is, from (6),

$$\omega'' = m\omega_c \cos \omega_m t. \quad (32)$$

Assuming the initial phase angle b_0 of each carrier is zero in (9), the modulated phase angle is

$$b'' = \int_0^t \omega'' dt = \frac{m\omega_c}{\omega_m} \sin \omega_m t. \quad (33)$$

From (28), the output of the balanced detector is

$$\frac{E_4}{1+k^2} = \frac{k^2}{1+k^2} m \cos \omega_m t \qquad \text{crosstalk}$$

$$+ \frac{k}{1+k^2} m \cos \omega_m t \cos b'' \qquad \text{beatnote.} \quad (34)$$

This is plotted in Fig. 5 for several cases. The desired signal is not modulated. The crosstalk output is a replica of the undesired modulating signal. The beatnote output is expressed completely as follows:

$$km \cos \omega_m t \cos \left(\frac{m\omega_c}{\omega_m} \sin \omega_m t \right). \quad (35)$$

The beat frequency is equal to the frequency deviation given in (32), both its amplitude and its frequency being modulated at the modulation frequency. In wide-band modulation, a low-pass filter would cut off the beatnote during part of the time, while its amplitude and deviation are greatest, and the remaining beatnote interference would be weaker and more irregular. Even without the filter, it would be less audible. Therefore, while the audibility of the beatnote interference increases with increasing weak modulation, it tends to decrease with increasing strong modulation which causes the frequency deviation to go much beyond the audio-frequency range.

IX. The Derivation for Two Signals in a Linear Frequency Detector

The output of each rectifier in the linear frequency detector is the magnitude of $E_{3\pm}$ given in (24). The square of this magnitude, from (25), is

$$(E_{4\pm})^2 = |E_{3\pm}|^2 = \frac{1}{4}(1 \pm E_1')^2 \left[1 + k^2 \left(\frac{1 \pm E_1''}{1 \pm E_1'} \right)^2 + 2k \left(\frac{1 \pm E_1''}{1 + E_1'} \right) \cos (b'' - b') \right]. \quad (36)$$

Since the square root of the bracketed expression is difficult to interpret, it is expanded in a series, including only terms in the first powers of E_1' and E_1'', and up to the second power of k.

$$E_{4\pm} = \tfrac{1}{2}(1 \pm E_1') \left\{ 1 + k \left(\frac{1 \pm E_1''}{1 \pm E_1'} \right) \cos (b'' - b') + \frac{k^2}{2} \left(\frac{1 \pm E_1''}{1 \pm E_1'} \right)^2 [1 - \cos^2 (b'' - b')] + \cdots \right\}$$

$$= \frac{1}{2}(1 \pm E_1') + \frac{k}{2} (1 \pm E_1'') \cos (b'' - b')$$

$$+ \frac{k^2}{8} (1 \mp E_1' \pm 2E_1'' \mp 2E_1'E_1'') [1 - \cos (2b'' - b')] + \cdots. \quad (37)$$

The differential output is

$$E_4 = E_{4+} - E_{4-}$$

$$= E_1' \left(1 - \frac{k^2}{4} + \cdots \right) \qquad \text{desired signal}$$

$$+ \frac{k^2}{2} E_1'' \qquad \text{crosstalk}$$

$$- \frac{k^2}{2} E_1'E_1'' \qquad \text{mixed signal}$$

$$+ \frac{k}{2} E_1'' \cos (b'' - b') \qquad \text{beatnote}$$

$$+ \frac{k^2}{4} (E_1' - 2E_1'' + 2E_1'E_1'' - \cdots) \cos (2b'' - b') + \cdots. \quad (38)$$

Two cases of this expansion appear as (3) and (4). It is complicated by the masking factor on the desired signal, and by the distortion (harmonics and intermodulation) of both signals. Each coefficient is part of an infinite series.

The automatic volume control in this case is also assumed to have a linear rectifier, to correspond with the detector. This control is substituted for the limiter L in Fig. 1, and is assumed to maintain the average value of the voltage E_3 at a uniform value of unity. Rewriting (23), the composite signal voltage is

$$E_2 = \exp ib' [1 + k \exp i(b'' - b')] \quad (39)$$

and its magnitude is

$$|E_2| = \sqrt{1 + k^2 + 2k \cos (b'' - b')}$$

$$= 1 + 2k \cos (b'' - b') + \frac{k^2}{4} [1 - \cos 2(b'' - b')] + \cdots. \quad (40)$$

The average value of its magnitude is

$$\overline{E_2} = \frac{2}{\pi}[2E(k) - (1 - k^2)K(k)] = 1 + \frac{k^2}{4} + \cdots$$

$$= \frac{4}{\pi} \quad (k = 1)$$

$$= k \quad (k \gg 1) \quad (41)$$

in which $K(k)$ and $E(k)$ are respectively the complete elliptic integrals of the first and second kinds.[11-13] The

[11] (The real part of K or E is used for $k > 1$.)
[12] Jahnke and Emde, "Tables of Functions," 1933, chapter 15, pp. 127, 145, 150.
[13] D. Bierens de Haan, "New Tables of Definite Integrals," 1867–1939. Table 67, (5) and (7); uses F' and E' instead of K and E.

value of E_3 becomes

$$E_3 = \frac{E_2}{\overline{E_2}} \tag{42}$$

which has an average value of unity.

The desired-signal output is reduced by two factors, the masking factor (in parentheses) (38), and the factor $1/\overline{E_2}$. Both factors cause more reduction with increasing relative amplitude k of the undesired signal.

A precise evaluation of the differential output E_4 also requires elliptic integrals. Instead of expanding into a series, formula (36) may be rewritten,

$$E_{4\pm} = \tfrac{1}{2}(1 \pm E_1')\sqrt{1 + k_\pm{}^2 + 2k_\pm \cos(b''-b')} \tag{43}$$

in which

$$k_\pm = k\,\frac{1 \pm E_1''}{1 \pm E_1'}\,. \tag{44}$$

In order to remove the beatnote, $E_{4\pm}$ is averaged over the beatnote cycle of $(b''-b')$, leaving only the modulating voltages as they appear in the output of either rectifier.

$$\overline{E_{4\pm}} = \frac{1}{2}(1 \pm E_1')\,\frac{2}{\pi}\,[2E(k_\pm) - (1-k_\pm{}^2)K(k_\pm)]. \tag{45}$$

This is obtained by the same elliptic integral as used to obtain (41) from (40) above. The differential output $\overline{E_4}$ is to be expressed only for special cases.

The distortion of the modulating voltages E_1' and E_1'' in the output complicates the solution, but adds little of interest because the more important effects involve the beatnote noise and the signal amplitudes. The latter information is obtained with close approximation by solving for the case of small modulating voltages:

$E_1' \ll 1,\ E_2' \ll 1,$

$$k_\pm = k, \qquad \frac{dk_\pm}{dE_1'} = \mp k, \qquad \frac{dk_\pm}{dE_1''} = \pm k. \tag{46}$$

The relation is used,[14]

$$\begin{aligned} \frac{d}{dk}\cdot\frac{2}{\pi}\,[2E(k)-(1-k^2)K(k)] &= \frac{2}{\pi k}\,[E(k)-(1-k^2)K(k)] \\ &= \frac{k}{2} \quad (k \ll 1) \\ &= \frac{2}{\pi} \quad (k = 1) \\ &= 1 \quad (k \ll 1). \end{aligned} \tag{47}$$

On this basis, the differential output of the detector (omitting the beatnote terms) is

[14] Jahnke and Emde, footnote reference 12, pp. 128–129.

$$\begin{aligned} \overline{E_4} = \overline{E_{4+}} - \overline{E_{4-}} &= \left(E_1'\frac{d}{dE_1'} + E_1''\frac{d}{dE_1''}\right)(E_{4+} - E_{4-}) \\ &= E_1'\,\frac{2}{\pi}\,E(k) \quad \text{desired signal} \\ &\quad + E_1''\,\frac{2}{\pi}\,[E(k) - (1-k^2)K(k)] \quad \text{crosstalk.} \end{aligned} \tag{48}$$

Applying the automatic-volume-control factor, the resulting output is

$$\begin{aligned} \frac{\overline{E_4}}{\overline{E_2}} &= E_1'\left(1 - \frac{E(k)-(1-k^2)K(k)}{2E(k)-(1-k^2)K(k)}\right) \quad \text{desired signal} \\ &\quad + E_1''\,\frac{E(k) - (1-k^2)K(k)}{2E(k) - (1-k^2)K(k)} \quad \text{crosstalk.} \end{aligned} \tag{49}$$

This formula shows that the reduction of response to the desired signal is equal to the increase of response to the undesired signal. This crosstalk is plotted as Fig. 13(b).

In a like manner, the beatnote fundamental and harmonic components can be evaluated in terms of elliptic integrals. For present purposes, however, the series expansion (38) is more useful.

X. The Derivation for Two Signals in a Limiter

The limiter L in Fig. 1 receives the composite signal voltage E_2 of varying amplitude. The action of the limiter on this voltage is such that its output voltage retains the instantaneous frequency of the input voltage but has a uniform amplitude of unity. This action is conceived as a fast-acting control of amplification which holds the signal envelope at a constant amplitude.

The composite input voltage is

$$\begin{aligned} E_2 = E_2' + E_2'' &= \exp ib' + k \exp ib'' \\ &= \exp ib'[1 + k \exp i(b''-b')] \\ &= \exp ib'[1 + k\cos(b''-b') + ik \sin(b''-b')] \\ &= |E_2| \exp ib = |E_2| \exp ib' \exp i(b''-b') \end{aligned} \tag{50}$$

in which b is the progressive phase angle of the composite signal and $|E_2|$ is the magnitude of the envelope. This phase angle is determined by the above expressions:

$$\tan(b-b') = \frac{k \sin(b''-b')}{1 + k\cos(b''-b')} \tag{51}$$

$$b = b' + \text{antitan}\,\frac{k\sin(b''-b')}{1 + k\cos(b''-b')}\,. \tag{52}$$

Therefore the frequency of the composite signal is

$$\begin{aligned} \omega = \frac{db}{dt} &= \frac{db'}{dt} + \frac{k^2 + k\cos(b''-b')}{1+k^2+2k\cos(b''-b')}\left(\frac{db''}{dt} - \frac{db'}{dt}\right) \\ &= \omega' + k(\omega'' - \omega')\,\frac{k + \cos(b''-b')}{1 + k^2 + 2k\cos(b''-b')}\,. \end{aligned} \tag{53}$$

The output of the limiter is

$$E_3 = \frac{E_2}{|E_2|} = \exp ib \tag{54}$$

which is a signal of unit amplitude and of phase b or frequency ω.

While this signal could be subjected to the processes in a frequency detector, this is unnecessary, because the detector in this case would merely deliver an output voltage proportional to the frequency (deviation). Therefore the differential output of the detector is

$$E_4 = \frac{\omega}{\omega_c} \cdot \tag{55}$$

This was proved for a signal of unit amplitude and any frequency modulation in the case of one signal; it is a conclusion from (17) and (19) in that case. In this case, the detector output is

$$E_4 = E_1' \quad \text{desired signal}$$
$$+k(E''-E')\frac{k+\cos(b''-b')}{1+k^2+2k\cos(b''-b')} \quad \text{beatnote.} \tag{56}$$

If the undesired signal is the weaker, $k<1$, the second term has an average value of zero, and therefore represents only the beatnote (fundamental and harmonics) as indicated. On the other hand, if the undesired signal is the stronger, $k>1$, it amounts to interchanging the two signals and inverting k:

$$E_4 = E_1'' \quad \text{crosstalk}$$
$$+\frac{1}{k}(E_1'-E_1'')\frac{\frac{1}{k}+\cos(b'-b'')}{1+\frac{1}{k^2}+\frac{2}{k}\cos(b'-b'')} \quad \text{beatnote.} \tag{57}$$

These two solutions show clearly the effect of the limiter in favoring the stronger of the two signals and eliminating the other, as plotted in Fig. 11(*a*) and (*b*). The beatnote is a harmonic series of the following form for $k<1$ (and corresponding form[15–17] for $1/k<1$):

$$k\frac{k+\cos(b''-b')}{1+k^2+2k\cos(b''-b')} = k\cos(b''-b')$$
$$-k^2\cos 2(b''-b') + k^3\cos 3(b''-b') - \cdots. \tag{58}$$

This beatnote series is included in (2) and the waveform is plotted in Fig. 4(c) on a scale which leaves the fundamental component the same for various values of k.

[15] Crosby (footnote 4) in equation (7) and Fig. 4 shows the harmonic waveform of the beatnote.

[16] E. P. Adams, "Smithsonian Mathematical Formulae," 1922. On p. 82, no. 13 is the expansion needed to express the beatnote in a Fourier series.

[17] de Haan's table 50, item (5), is the integral form needed to evaluate the coefficients of the beatnote Fourier series.

The beatnote oscillates between the peak values

$$(k<1)\ \frac{k}{1+k} \text{ and } -\frac{k}{1-k};\quad (k>1)\ \frac{1}{k+1} \text{ and } -\frac{1}{k-1}\cdot \tag{59}$$

Its quadratic-mean (root-mean-square) value is

$$=\sqrt{\tfrac{1}{2}(k^2+k^4+k^6+\cdots)} = \frac{k}{\sqrt{2(1-k^2)}} \quad (k<1). \tag{60}$$

An interesting example is that of no modulation of the desired signal and sinusoidal modulation of a weaker undesired signal. Only the beatnote term remains:

$$E_4 = kE_1''\frac{k+\cos b''}{1+k^2+2k\cos b''} \quad (k<1) \tag{61}$$

in which

$$b'' = \frac{m\omega_c}{\omega_m}\sin\omega_m t;\qquad \omega'' = \frac{db''}{dt} = m\omega_c\cos\omega_m t. \tag{62}$$

In these formulas, $m\omega_c$ is the maximum deviation, ω_m is the modulating frequency, and ω'' is the beat frequency. This output voltage E_4 is plotted in Fig. 6.

XI. Conclusion

There are several types of balanced frequency detectors capable of reproducing without distortion a frequency-modulated signal. Two of these types employ linear slope filters, one with linear rectifiers and the other with square-law rectifiers.

If a perfect limiter is assumed, it is immaterial which type of frequency detector is used; otherwise the choice depends on which type has the more desirable behavior toward interference. Toward common-channel interference, the linear rectifiers give less crosstalk from the weaker of the two signals. During modulation of the stronger signal, the linear rectifiers give less beatnote interference.

The different kinds of interference are subdivided into frequency and amplitude effects. The frequency effects are inherently associated with the ability to detect the frequency modulation of the signals. The amplitude effects can be avoided by the use of a limiter; they are twice as great with square-law rectifiers as with linear rectifiers.

The beatnote interference is caused by frequency and amplitude modulation in the composite signal. Its interference effect is mainly caused by the frequency modulation, so it cannot be reduced very much by avoiding the amplitude effects. It can be reduced by using a bandwidth of frequency modulation exceeding twice the bandwidth required by the modulating signal.

The crosstalk interference from a weaker signal is caused entirely by the amplitude modulation in the composite signal. It can be minimized by the use of a limiter, and its amplitude from linear rectifiers is only one half as great as from square-law rectifiers.

The conical pattern, especially well adapted for oscilloscope tests, is useful in observing common-channel interference and in comparing the frequency modulation of one signal with the steady frequency of another.

The relations described have been verified by tests. The waveform of the interference output has been found to agree with Figs. 5 and 6, in corresponding cases without and with a limiter. The conical diagrams of interference output, Figs. 7 and 8, have been reproduced on the oscilloscope under the various conditions illustrated. The conical diagram of Fig. 9 has been used in checking the amount of frequency modulation in a signal. The crosstalk component has been separated from the beatnote by a filter to test the relations of Fig. 13, both without a limiter and with limiters of practical design. The beatnote amplitude, as in Figs. 11 and 12, has been checked by oscilloscope observations corresponding to Figs. 5 to 8. The nature of the crosstalk and beatnote interference has been verified in listening tests.